9.2 Series and Convergence

- Understand the definition of a convergent infinite series.
- Use properties of infinite geometric series.
- Use the *n*th-Term Test for Divergence of an infinite series.

Infinite Series

One important application of infinite sequences is in representing "infinite summations." Informally, if $\{a_n\}$ is an infinite sequence, then

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdot \cdot \cdot + a_n + \cdot \cdot \cdot$$

Infinite Series

is an **infinite series** (or simply a **series**). The numbers a_1, a_2, a_3, and so on are the **terms** of the series. For some series, it is convenient to begin the index at $n = 0$ (or some other integer). As a typesetting convention, it is common to represent an infinite series as $\Sigma\, a_n$. In such cases, the starting value for the index must be taken from the context of the statement.

To find the sum of an infinite series, consider the **sequence of partial sums** listed below.

$$\begin{aligned}
S_1 &= a_1 \\
S_2 &= a_1 + a_2 \\
S_3 &= a_1 + a_2 + a_3 \\
S_4 &= a_1 + a_2 + a_3 + a_4 \\
S_5 &= a_1 + a_2 + a_3 + a_4 + a_5 \\
&\vdots \\
S_n &= a_1 + a_2 + a_3 + \cdot \cdot \cdot + a_n
\end{aligned}$$

If this sequence of partial sums converges, then the series is said to converge and has the sum indicated in the next definition.

REMARK As you study this chapter, it is important to distinguish between an infinite series and a sequence. A sequence is an ordered collection of numbers

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

whereas a series is an infinite sum of terms from a sequence

$$a_1 + a_2 + a_3 + \cdot \cdot \cdot + a_n + \cdot \cdot \cdot.$$

Definitions of Convergent and Divergent Series

For the infinite series $\sum_{n=1}^{\infty} a_n$, the ***n*th partial sum** is

$$S_n = a_1 + a_2 + \cdot \cdot \cdot + a_n.$$

If the sequence of partial sums $\{S_n\}$ converges to S, then the series $\sum_{n=1}^{\infty} a_n$ **converges.** The limit S is called the **sum of the series.**

$$S = a_1 + a_2 + \cdot \cdot \cdot + a_n + \cdot \cdot \cdot \qquad S = \sum_{n=1}^{\infty} a_n$$

If $\{S_n\}$ diverges, then the series **diverges.**

INFINITE SERIES

The study of infinite series was considered a novelty in the fourteenth century. Logician Richard Suiseth, whose nickname was Calculator, solved this problem.

If throughout the first half of a given time interval a variation continues at a certain intensity, throughout the next quarter of the interval at double the intensity, throughout the following eighth at triple the intensity and so ad infinitum; then the average intensity for the whole interval will be the intensity of the variation during the second subinterval (or double the intensity).

This is the same as saying that the sum of the infinite series

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdot \cdot \cdot + \frac{n}{2^n} + \cdot \cdot \cdot$$

is 2.

As you study this chapter, you will see that there are two basic questions involving infinite series.

- Does a series converge or does it diverge?
- When a series converges, what is its sum?

These questions are not always easy to answer, especially the second one.

TECHNOLOGY Figure 9.5 shows the first 15 partial sums of the infinite series in Example 1(a). Notice how the values appear to approach the line $y = 1$.

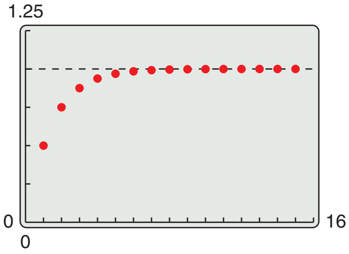


Figure 9.5

EXAMPLE 1 Convergent and Divergent Series

a. The series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

has the partial sums listed below. (You can also determine the partial sums of the series geometrically, as shown in Figure 9.6.)

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$\vdots$$

$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

Because

$$\lim_{n\to\infty} \frac{2^n - 1}{2^n} = 1$$

it follows that the series converges and its sum is 1.

b. The nth partial sum of the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots$$

is

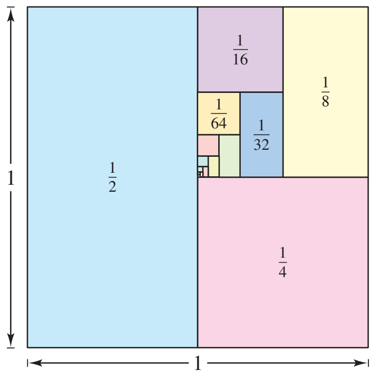
$$S_n = 1 - \frac{1}{n+1}.$$

Because the limit of S_n is 1, the series converges and its sum is 1.

c. The series

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \cdots$$

diverges because $S_n = n$ and the sequence of partial sums diverges.



You can determine the partial sums of the series in Example 1(a) geometrically using this figure.
Figure 9.6

The series in Example 1(b) is a **telescoping series** of the form

$$(b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + (b_4 - b_5) + \cdots. \quad \text{Telescoping series}$$

Note that b_2 is canceled by the second term, b_3 is canceled by the third term, and so on. Because the nth partial sum of this series is

$$S_n = b_1 - b_{n+1}$$

it follows that a telescoping series will converge if and only if b_n approaches a finite number as $n \to \infty$. Moreover, if the series converges, then its sum is

$$S = b_1 - \lim_{n\to\infty} b_{n+1}.$$

FOR FURTHER INFORMATION To learn more about the partial sums of infinite series, see the article "Six Ways to Sum a Series" by Dan Kalman in *The College Mathematics Journal*. To view this article, go to *MathArticles.com*.

EXAMPLE 2 **Writing a Series in Telescoping Form**

Find the sum of the series $\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}$.

Solution

Using partial fractions, you can write

$$a_n = \frac{2}{4n^2 - 1} = \frac{2}{(2n-1)(2n+1)} = \frac{1}{2n-1} - \frac{1}{2n+1}.$$

From this telescoping form, you can see that the nth partial sum is

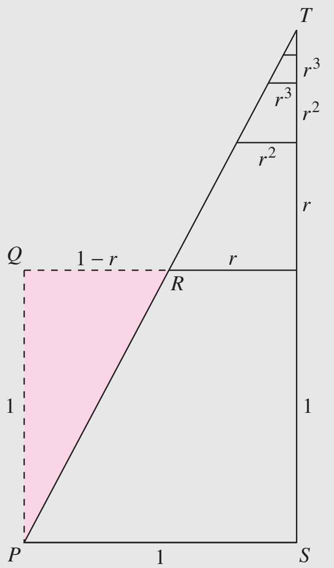
$$S_n = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdot \cdot \cdot + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = 1 - \frac{1}{2n+1}.$$

So, the series converges and its sum is 1. That is,

$$\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{2n+1}\right) = 1.$$

Exploration

In "Proof Without Words," by Benjamin G. Klein and Irl C. Bivens, the authors present the diagram below. Explain why the second statement after the diagram is valid. How is this result related to Theorem 9.6?



$\Delta PQR \sim \Delta TSP$

$$1 + r + r^2 + r^3 + \cdot \cdot \cdot = \frac{1}{1-r}$$

Exercise taken from "Proof Without Words" by Benjamin G. Klein and Irl C. Bivens, *Mathematics Magazine,* 61, No. 4, October 1988, p. 219, by permission of the authors.

Geometric Series

The series in Example 1(a) is a **geometric series.** In general, the series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdot \cdot \cdot + ar^n + \cdot \cdot \cdot, \quad a \neq 0$$

Geometric series

is a **geometric series** with ratio r, $r \neq 0$.

THEOREM 9.6 **Convergence of a Geometric Series**

A geometric series with ratio r diverges when $|r| \geq 1$. If $0 < |r| < 1$, then the series converges to the sum

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad 0 < |r| < 1.$$

Proof It is easy to see that the series diverges when $r = \pm 1$. If $r \neq \pm 1$, then

$$S_n = a + ar + ar^2 + \cdot \cdot \cdot + ar^{n-1}.$$

Multiplication by r yields

$$rS_n = ar + ar^2 + ar^3 + \cdot \cdot \cdot + ar^n.$$

Subtracting the second equation from the first produces $S_n - rS_n = a - ar^n$. Therefore, $S_n(1 - r) = a(1 - r^n)$, and the nth partial sum is

$$S_n = \frac{a}{1-r}(1 - r^n).$$

When $0 < |r| < 1$, it follows that $r^n \to 0$ as $n \to \infty$, and you obtain

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left[\frac{a}{1-r}(1 - r^n)\right] = \frac{a}{1-r}\left[\lim_{n\to\infty} (1 - r^n)\right] = \frac{a}{1-r}$$

which means that the series *converges* and its sum is $a/(1 - r)$. It is left to you to show that the series diverges when $|r| > 1$.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

TECHNOLOGY Try using a graphing utility to compute the sum of the first 20 terms of the sequence in Example 3(a). You should obtain a sum of about 5.999994.

EXAMPLE 3 Convergent and Divergent Geometric Series

a. The geometric series

$$\sum_{n=0}^{\infty} \frac{3}{2^n} = \sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n = 3(1) + 3\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \cdots$$

has a ratio of $r = \frac{1}{2}$ with $a = 3$. Because $0 < |r| < 1$, the series converges and its sum is

$$S = \frac{a}{1-r} = \frac{3}{1-(1/2)} = 6.$$

b. The geometric series

$$\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n = 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \cdots$$

has a ratio of $r = \frac{3}{2}$. Because $|r| \geq 1$, the series diverges.

The formula for the sum of a geometric series can be used to write a repeating decimal as the ratio of two integers, as demonstrated in the next example.

EXAMPLE 4 A Geometric Series for a Repeating Decimal

See LarsonCalculus.com for an interactive version of this type of example.

Use a geometric series to write $0.\overline{08}$ as the ratio of two integers.

Solution For the repeating decimal $0.\overline{08}$, you can write

$$\begin{aligned} 0.080808\ldots &= \frac{8}{10^2} + \frac{8}{10^4} + \frac{8}{10^6} + \frac{8}{10^8} + \cdots \\ &= \sum_{n=0}^{\infty} \left(\frac{8}{10^2}\right)\left(\frac{1}{10^2}\right)^n. \end{aligned}$$

For this series, you have $a = 8/10^2$ and $r = 1/10^2$. So,

$$0.080808\ldots = \frac{a}{1-r} = \frac{8/10^2}{1-(1/10^2)} = \frac{8}{99}.$$

Try dividing 8 by 99 on a calculator to see that it produces $0.\overline{08}$.

The convergence of a series is not affected by the removal of a finite number of terms from the beginning of the series. For instance, the geometric series

$$\sum_{n=4}^{\infty} \left(\frac{1}{2}\right)^n \quad \text{and} \quad \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

both converge. Furthermore, because the sum of the second series is

$$\frac{a}{1-r} = \frac{1}{1-(1/2)} = 2$$

you can conclude that the sum of the first series is

$$\begin{aligned} S &= 2 - \left[\left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3\right] \\ &= 2 - \frac{15}{8} \\ &= \frac{1}{8}. \end{aligned}$$

The properties in the next theorem are direct consequences of the corresponding properties of limits of sequences.

THEOREM 9.7 Properties of Infinite Series

Let $\Sigma\, a_n$ and $\Sigma\, b_n$ be convergent series, and let A, B, and c be real numbers. If $\Sigma\, a_n = A$ and $\Sigma\, b_n = B$, then the following series converge to the indicated sums.

1. $\displaystyle\sum_{n=1}^{\infty} ca_n = cA$
2. $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = A + B$
3. $\displaystyle\sum_{n=1}^{\infty} (a_n - b_n) = A - B$

*n*th-Term Test for Divergence

The next theorem states that when a series converges, the limit of its nth term must be 0.

REMARK Be sure you see that the converse of Theorem 9.8 is generally not true. That is, if the sequence $\{a_n\}$ converges to 0, then the series $\Sigma\, a_n$ may either converge or diverge.

THEOREM 9.8 Limit of the *n*th Term of a Convergent Series

If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\lim_{n\to\infty} a_n = 0$.

Proof Assume that

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = L.$$

Then, because $S_n = S_{n-1} + a_n$ and

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1} = L$$

it follows that

$$\begin{aligned} L &= \lim_{n\to\infty} S_n \\ &= \lim_{n\to\infty} (S_{n-1} + a_n) \\ &= \lim_{n\to\infty} S_{n-1} + \lim_{n\to\infty} a_n \\ &= L + \lim_{n\to\infty} a_n \end{aligned}$$

which implies that $\{a_n\}$ converges to 0.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

The contrapositive of Theorem 9.8 provides a useful test for *divergence*. This ***n*th-Term Test for Divergence** states that if the limit of the nth term of a series does *not* converge to 0, then the series must diverge.

THEOREM 9.9 *n*th-Term Test for Divergence

If $\displaystyle\lim_{n\to\infty} a_n \neq 0$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

EXAMPLE 5 Using the *n*th-Term Test for Divergence

a. For the series $\sum_{n=0}^{\infty} 2^n$, you have

$$\lim_{n\to\infty} 2^n = \infty.$$

So, the limit of the nth term is not 0, and the series diverges.

b. For the series $\sum_{n=1}^{\infty} \frac{n!}{2n! + 1}$, you have

$$\lim_{n\to\infty} \frac{n!}{2n! + 1} = \frac{1}{2}.$$

So, the limit of the nth term is not 0, and the series diverges.

c. For the series $\sum_{n=1}^{\infty} \frac{1}{n}$, you have

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Because the limit of the nth term is 0, the nth-Term Test for Divergence does *not* apply and you can draw no conclusions about convergence or divergence. (In the next section, you will see that this particular series diverges.)

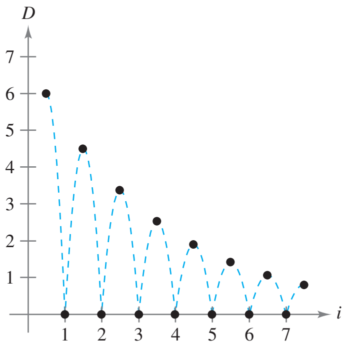
REMARK The series in Example 5(c) will play an important role in this chapter.

$$\sum_{n=1}^{\infty} \frac{1}{n} =$$
$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

You will see that this series diverges even though the nth term approaches 0 as n approaches ∞.

EXAMPLE 6 Bouncing Ball Problem

A ball is dropped from a height of 6 feet and begins bouncing, as shown in Figure 9.7. The height of each bounce is three-fourths the height of the previous bounce. Find the total vertical distance traveled by the ball.



The height of each bounce is three-fourths the height of the preceding bounce.

Figure 9.7

Solution When the ball hits the ground for the first time, it has traveled a distance of $D_1 = 6$ feet. For subsequent bounces, let D_i be the distance traveled up and down. For example, D_2 and D_3 are

$$D_2 = \underbrace{6\left(\frac{3}{4}\right)}_{\text{Up}} + \underbrace{6\left(\frac{3}{4}\right)}_{\text{Down}} = 12\left(\frac{3}{4}\right)$$

and

$$D_3 = \underbrace{6\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)}_{\text{Up}} + \underbrace{6\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)}_{\text{Down}} = 12\left(\frac{3}{4}\right)^2.$$

By continuing this process, it can be determined that the total vertical distance is

$$\begin{aligned}
D &= 6 + 12\left(\frac{3}{4}\right) + 12\left(\frac{3}{4}\right)^2 + 12\left(\frac{3}{4}\right)^3 + \cdots \\
&= 6 + 12\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^{n+1} \\
&= 6 + 12\left(\frac{3}{4}\right)\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^{n} \\
&= 6 + 9\left[\frac{1}{1 - (3/4)}\right] \\
&= 6 + 9(4) \\
&= 42 \text{ feet.}
\end{aligned}$$

9.2 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Finding Partial Sums In Exercises 1–6, find the sequence of partial sums S_1, S_2, S_3, S_4, and S_5.

1. $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$
2. $\frac{1}{2 \cdot 3} + \frac{2}{3 \cdot 4} + \frac{3}{4 \cdot 5} + \frac{4}{5 \cdot 6} + \frac{5}{6 \cdot 7} + \cdots$
3. $3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} + \frac{243}{16} - \cdots$
4. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \cdots$
5. $\sum_{n=1}^{\infty} \frac{3}{2^{n-1}}$
6. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$

Verifying Divergence In Exercises 7–14, verify that the infinite series diverges.

7. $\sum_{n=0}^{\infty} \left(\frac{7}{6}\right)^n$
8. $\sum_{n=0}^{\infty} 4(-1.05)^n$
9. $\sum_{n=1}^{\infty} \frac{n}{n+1}$
10. $\sum_{n=1}^{\infty} \frac{n}{2n+3}$
11. $\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$
12. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$
13. $\sum_{n=1}^{\infty} \frac{2^n+1}{2^{n+1}}$
14. $\sum_{n=1}^{\infty} \frac{n!}{2^n}$

Verifying Convergence In Exercises 15–20, verify that the infinite series converges.

15. $\sum_{n=0}^{\infty} \left(\frac{5}{6}\right)^n$
16. $\sum_{n=1}^{\infty} 2\left(-\frac{1}{2}\right)^n$
17. $\sum_{n=0}^{\infty} (0.9)^n = 1 + 0.9 + 0.81 + 0.729 + \cdots$
18. $\sum_{n=0}^{\infty} (-0.6)^n = 1 - 0.6 + 0.36 - 0.216 + \cdots$
19. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ (*Hint:* Use partial fractions.)
20. $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ (*Hint:* Use partial fractions.)

Numerical, Graphical, and Analytic Analysis In Exercises 21–24, (a) find the sum of the series, (b) use a graphing utility to find the indicated partial sum S_n and complete the table, (c) use a graphing utility to graph the first 10 terms of the sequence of partial sums and a horizontal line representing the sum, and (d) explain the relationship between the magnitudes of the terms of the series and the rate at which the sequence of partial sums approaches the sum of the series.

n	5	10	20	50	100
S_n					

21. $\sum_{n=1}^{\infty} \frac{6}{n(n+3)}$
22. $\sum_{n=1}^{\infty} \frac{4}{n(n+4)}$
23. $\sum_{n=1}^{\infty} 2(0.9)^{n-1}$
24. $\sum_{n=1}^{\infty} 10\left(-\frac{1}{4}\right)^{n-1}$

Finding the Sum of a Convergent Series In Exercises 25–34, find the sum of the convergent series.

25. $\sum_{n=0}^{\infty} 5\left(\frac{2}{3}\right)^n$
26. $\sum_{n=0}^{\infty} \left(-\frac{1}{5}\right)^n$
27. $\sum_{n=1}^{\infty} \frac{4}{n(n+2)}$
28. $\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)}$
29. $8 + 6 + \frac{9}{2} + \frac{27}{8} + \cdots$
30. $9 - 3 + 1 - \frac{1}{3} + \cdots$
31. $\sum_{n=0}^{\infty} \left(\frac{1}{2^n} - \frac{1}{3^n}\right)$
32. $\sum_{n=0}^{\infty} [(0.3)^n + (0.8)^n]$
33. $\sum_{n=1}^{\infty} (\sin 1)^n$
34. $\sum_{n=1}^{\infty} \frac{1}{9n^2 + 3n - 2}$

Using a Geometric Series In Exercises 35–40, (a) write the repeating decimal as a geometric series, and (b) write its sum as the ratio of two integers.

35. $0.\overline{4}$
36. $0.\overline{36}$
37. $0.\overline{81}$
38. $0.\overline{01}$
39. $0.0\overline{75}$
40. $0.2\overline{15}$

Determining Convergence or Divergence In Exercises 41–54, determine the convergence or divergence of the series.

41. $\sum_{n=0}^{\infty} (1.075)^n$
42. $\sum_{n=0}^{\infty} \frac{3^n}{1000}$
43. $\sum_{n=1}^{\infty} \frac{n+10}{10n+1}$
44. $\sum_{n=1}^{\infty} \frac{4n+1}{3n-1}$
45. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$
46. $\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$
47. $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$
48. $\sum_{n=0}^{\infty} \frac{3}{5^n}$
49. $\sum_{n=2}^{\infty} \frac{n}{\ln n}$
50. $\sum_{n=1}^{\infty} \ln \frac{1}{n}$
51. $\sum_{n=1}^{\infty} \left(1 + \frac{k}{n}\right)^n$
52. $\sum_{n=1}^{\infty} e^{-n}$
53. $\sum_{n=1}^{\infty} \arctan n$
54. $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

WRITING ABOUT CONCEPTS

55. **Series** State the definitions of convergent and divergent series.
56. **Sequence and Series** Describe the difference between $\lim_{n\to\infty} a_n = 5$ and $\sum_{n=1}^{\infty} a_n = 5$.

WRITING ABOUT CONCEPTS (continued)

57. Geometric Series Define a geometric series, state when it converges, and give the formula for the sum of a convergent geometric series.

58. *n*th-Term Test for Divergence State the nth-Term Test for Divergence.

59. Comparing Series Explain any differences among the following series.

(a) $\sum_{n=1}^{\infty} a_n$ (b) $\sum_{k=1}^{\infty} a_k$ (c) $\sum_{n=1}^{\infty} a_k$

60. Using a Series

(a) You delete a finite number of terms from a divergent series. Will the new series still diverge? Explain your reasoning.

(b) You add a finite number of terms to a convergent series. Will the new series still converge? Explain your reasoning.

Making a Series Converge **In Exercises 61–66, find all values of x for which the series converges. For these values of x, write the sum of the series as a function of x.**

61. $\sum_{n=1}^{\infty} (3x)^n$

62. $\sum_{n=0}^{\infty} \left(\frac{2}{x}\right)^n$

63. $\sum_{n=1}^{\infty} (x-1)^n$

64. $\sum_{n=0}^{\infty} 5\left(\frac{x-2}{3}\right)^n$

65. $\sum_{n=0}^{\infty} (-1)^n x^n$

66. $\sum_{n=0}^{\infty} (-1)^n x^{2n}$

Using a Geometric Series **In Exercises 67 and 68, (a) find the common ratio of the geometric series, (b) write the function that gives the sum of the series, and (c) use a graphing utility to graph the function and the partial sums S_3 and S_5. What do you notice?**

67. $1 + x + x^2 + x^3 + \cdots$

68. $1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots$

Writing **In Exercises 69 and 70, use a graphing utility to determine the first term that is less than 0.0001 in each of the convergent series. Note that the answers are very different. Explain how this will affect the rate at which the series converges.**

69. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}, \quad \sum_{n=1}^{\infty} \left(\frac{1}{8}\right)^n$

70. $\sum_{n=1}^{\infty} \frac{1}{2^n}, \quad \sum_{n=1}^{\infty} (0.01)^n$

71. Marketing An electronic games manufacturer producing a new product estimates the annual sales to be 8000 units. Each year, 5% of the units that have been sold will become inoperative. So, 8000 units will be in use after 1 year, $[8000 + 0.95(8000)]$ units will be in use after 2 years, and so on. How many units will be in use after n years?

72. Depreciation A company buys a machine for \$475,000 that depreciates at a rate of 30% per year. Find a formula for the value of the machine after n years. What is its value after 5 years?

73. Multiplier Effect

The total annual spending by tourists in a resort city is \$200 million. Approximately 75% of that revenue is again spent in the resort city, and of that amount approximately 75% is again spent in the same city, and so on. Write the geometric series that gives the total amount of spending generated by the \$200 million and find the sum of the series.

AISPIX by Image Source/Shutterstock.com

74. Multiplier Effect Repeat Exercise 73 when the percent of the revenue that is spent again in the city decreases to 60%.

75. Distance A ball is dropped from a height of 16 feet. Each time it drops h feet, it rebounds $0.81h$ feet. Find the total distance traveled by the ball.

76. Time The ball in Exercise 75 takes the following times for each fall.

$$\begin{aligned}
s_1 &= -16t^2 + 16, & s_1 &= 0 \text{ when } t = 1\\
s_2 &= -16t^2 + 16(0.81), & s_2 &= 0 \text{ when } t = 0.9\\
s_3 &= -16t^2 + 16(0.81)^2, & s_3 &= 0 \text{ when } t = (0.9)^2\\
s_4 &= -16t^2 + 16(0.81)^3, & s_4 &= 0 \text{ when } t = (0.9)^3\\
&\vdots & &\vdots\\
s_n &= -16t^2 + 16(0.81)^{n-1}, & s_n &= 0 \text{ when } t = (0.9)^{n-1}
\end{aligned}$$

Beginning with s_2, the ball takes the same amount of time to bounce up as it does to fall, and so the total time elapsed before it comes to rest is given by

$$t = 1 + 2\sum_{n=1}^{\infty} (0.9)^n.$$

Find this total time.

Probability **In Exercises 77 and 78, the random variable n represents the number of units of a product sold per day in a store. The probability distribution of n is given by $P(n)$. Find the probability that two units are sold in a given day $[P(2)]$ and show that $P(0) + P(1) + P(2) + P(3) + \cdots = 1$.**

77. $P(n) = \frac{1}{2}\left(\frac{1}{2}\right)^n$

78. $P(n) = \frac{1}{3}\left(\frac{2}{3}\right)^n$

79. Probability A fair coin is tossed repeatedly. The probability that the first head occurs on the nth toss is given by $P(n) = \left(\frac{1}{2}\right)^n$, where $n \geq 1$.

(a) Show that $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1$.

(b) The expected number of tosses required until the first head occurs in the experiment is given by

$$\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n.$$

Is this series geometric?

(c) Use a computer algebra system to find the sum in part (b).

80. Probability In an experiment, three people toss a fair coin one at a time until one of them tosses a head. Determine, for each person, the probability that he or she tosses the first head. Verify that the sum of the three probabilities is 1.

81. Area The sides of a square are 16 inches in length. A new square is formed by connecting the midpoints of the sides of the original square, and two of the triangles outside the second square are shaded (see figure). Determine the area of the shaded regions (a) when this process is continued five more times, and (b) when this pattern of shading is continued infinitely.

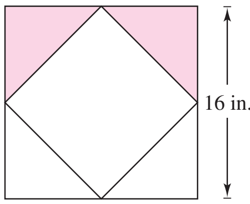


Figure for 81

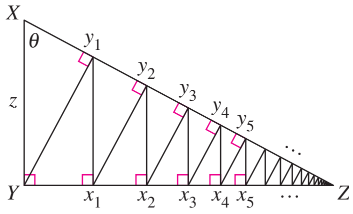


Figure for 82

82. Length A right triangle XYZ is shown above where $|XY| = z$ and $\angle X = \theta$. Line segments are continually drawn to be perpendicular to the triangle, as shown in the figure.

(a) Find the total length of the perpendicular line segments $|Yy_1| + |x_1y_1| + |x_1y_2| + \cdots$ in terms of z and θ.

(b) Find the total length of the perpendicular line segments when $z = 1$ and $\theta = \pi/6$.

Using a Geometric Series **In Exercises 83–86, use the formula for the nth partial sum of a geometric series**

$$\sum_{i=0}^{n-1} ar^i = \frac{a(1-r^n)}{1-r}.$$

83. Present Value The winner of a \$2,000,000 sweepstakes will be paid \$100,000 per year for 20 years. The money earns 6% interest per year. The present value of the winnings is $\sum_{n=1}^{20} 100{,}000\left(\frac{1}{1.06}\right)^n$. Compute the present value and interpret its meaning.

84. Annuities When an employee receives a paycheck at the end of each month, P dollars is invested in a retirement account. These deposits are made each month for t years and the account earns interest at the annual percentage rate r. When the interest is compounded monthly, the amount A in the account at the end of t years is

$$A = P + P\left(1 + \frac{r}{12}\right) + \cdots + P\left(1 + \frac{r}{12}\right)^{12t-1}$$
$$= P\left(\frac{12}{r}\right)\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right].$$

When the interest is compounded continuously, the amount A in the account after t years is

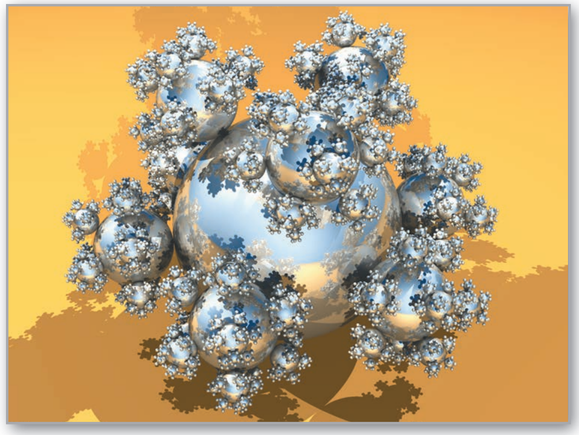
$$A = P + Pe^{r/12} + Pe^{2r/12} + Pe^{(12t-1)r/12}$$
$$= \frac{P(e^{rt} - 1)}{e^{r/12} - 1}.$$

Verify the formulas for the sums given above.

85. Salary You go to work at a company that pays \$0.01 for the first day, \$0.02 for the second day, \$0.04 for the third day, and so on. If the daily wage keeps doubling, what would your total income be for working (a) 29 days, (b) 30 days, and (c) 31 days?

86. Sphereflake

The sphereflake shown below is a computer-generated fractal that was created by Eric Haines. The radius of the large sphere is 1. To the large sphere, nine spheres of radius $\frac{1}{3}$ are attached. To each of these, nine spheres of radius $\frac{1}{9}$ are attached. This process is continued infinitely. Prove that the sphereflake has an infinite surface area.



Annuities **In Exercises 87–90, consider making monthly deposits of P dollars in a savings account at an annual interest rate r. Use the results of Exercise 84 to find the balance A after t years when the interest is compounded (a) monthly and (b) continuously.**

87. $P = \$45, \quad r = 3\%, \quad t = 20$ years

88. $P = \$75, \quad r = 5.5\%, \quad t = 25$ years

89. $P = \$100, \quad r = 4\%, \quad t = 35$ years

90. $P = \$30, \quad r = 6\%, \quad t = 50$ years

True or False? **In Exercises 91–96, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

91. If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

92. If $\sum_{n=1}^{\infty} a_n = L$, then $\sum_{n=0}^{\infty} a_n = L + a_0$.

93. If $|r| < 1$, then $\sum_{n=1}^{\infty} ar^n = \frac{a}{1-r}$.

94. The series $\sum_{n=1}^{\infty} \frac{n}{1000(n+1)}$ diverges.

95. $0.75 = 0.749999\ldots$

96. Every decimal with a repeating pattern of digits is a rational number.

Courtesy of Eric Haines

97. Using Divergent Series Find two divergent series $\Sigma\, a_n$ and $\Sigma\, b_n$ such that $\Sigma(a_n + b_n)$ converges.

98. Proof Given two infinite series $\Sigma\, a_n$ and $\Sigma\, b_n$ such that $\Sigma\, a_n$ converges and $\Sigma\, b_n$ diverges, prove that $\Sigma(a_n + b_n)$ diverges.

99. Fibonacci Sequence The Fibonacci sequence is defined recursively by $a_{n+2} = a_n + a_{n+1}$, where $a_1 = 1$ and $a_2 = 1$.

(a) Show that $\dfrac{1}{a_{n+1}\,a_{n+3}} = \dfrac{1}{a_{n+1}\,a_{n+2}} - \dfrac{1}{a_{n+2}\,a_{n+3}}$.

(b) Show that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{a_{n+1}\,a_{n+3}} = 1$.

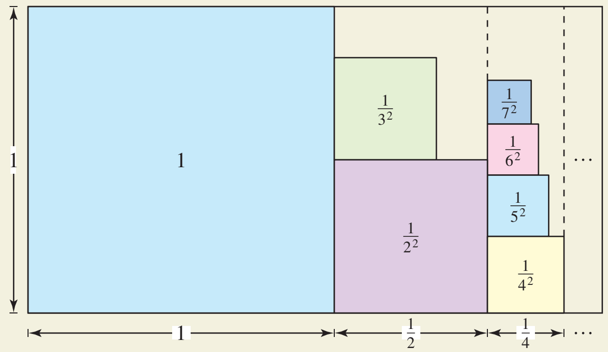
100. Remainder Let $\Sigma\, a_n$ be a convergent series, and let

$$R_N = a_{N+1} + a_{N+2} + \cdots$$

be the remainder of the series after the first N terms. Prove that $\displaystyle\lim_{N\to\infty} R_N = 0$.

101. Proof Prove that $\dfrac{1}{r} + \dfrac{1}{r^2} + \dfrac{1}{r^3} + \cdots = \dfrac{1}{r-1}$, for $|r| > 1$.

102. HOW DO YOU SEE IT? The figure below represents an informal way of showing that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} < 2$. Explain how the figure implies this conclusion.



■ **FOR FURTHER INFORMATION** For more on this exercise, see the article "Convergence with Pictures" by P. J. Rippon in *American Mathematical Monthly.*

PUTNAM EXAM CHALLENGE

103. Express $\displaystyle\sum_{k=1}^{\infty} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)}$ as a rational number.

104. Let $f(n)$ be the sum of the first n terms of the sequence 0, 1, 1, 2, 2, 3, 3, 4, . . . , where the nth term is given by

$$a_n = \begin{cases} n/2, & \text{if } n \text{ is even} \\ (n-1)/2, & \text{if } n \text{ is odd} \end{cases}.$$

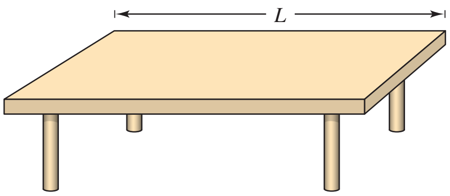
Show that if x and y are positive integers and $x > y$ then $xy = f(x + y) - f(x - y)$.

SECTION PROJECT

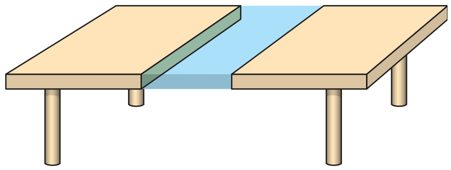
Cantor's Disappearing Table

The following procedure shows how to make a table disappear by removing only half of the table!

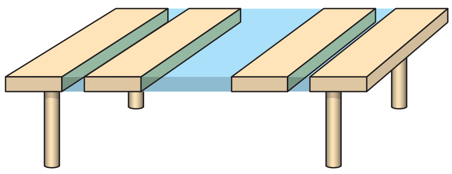
(a) Original table has a length of L.



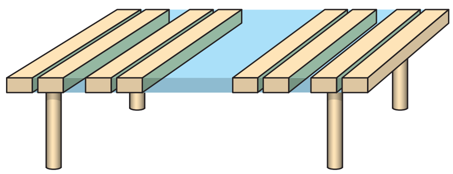
(b) Remove $\frac{1}{4}$ of the table centered at the midpoint. Each remaining piece has a length that is less than $\frac{1}{2}L$.



(c) Remove $\frac{1}{8}$ of the table by taking sections of length $\frac{1}{16}L$ from the centers of each of the two remaining pieces. Now, you have removed $\frac{1}{4} + \frac{1}{8}$ of the table. Each remaining piece has a length that is less than $\frac{1}{4}L$.



(d) Remove $\frac{1}{16}$ of the table by taking sections of length $\frac{1}{64}L$ from the centers of each of the four remaining pieces. Now, you have removed $\frac{1}{4} + \frac{1}{8} + \frac{1}{16}$ of the table. Each remaining piece has a length that is less than $\frac{1}{8}L$.



Will continuing this process cause the table to disappear, even though you have only removed half of the table? Why?

■ **FOR FURTHER INFORMATION** Read the article "Cantor's Disappearing Table" by Larry E. Knop in *The College Mathematics Journal.* To view this article, go to *MathArticles.com.*